For successful processing of ultrasonic information it is important to understand fully the physics behind the acquired signals. Having insight in the measurement situation makes it possible to extract features that are not readily seen from the data. The understanding of the physical laws also gives the possibility to create new measurement situations.

This chapter considers the fundamentals of medical ultrasound. It starts with the wave equation governing the propagation of ultrasound in human tissue. Then solutions to the wave equation are given for different coordinate systems. Most of the results in this dissertation are obtained through simulations using the program Field II [19]. The program is based on the calculation of spatial impulse responses and therefore this method for solving the wave equation is also considered. Further the approximations to the solution of the radiation problem known as the Fresnel and Fraunhofer approximations are presented to the reader, followed by an introduction to delay and sum beamforming. The chapter ends with the introduction of the frequency domain representation of the ultrasound systems, known also as $k$-space representation.

3.1 The wave equation

The opening paragraph is drawn from Insana and Brown [20]:

Fluids have elasticity (compressibility $\kappa$) and inertia (mass density $\rho$), the two characteristics required for wave phenomena in a spatially distributed physical system whose elements are coupled. Elasticity implies that any deviation from the equilibrium state of the fluid will tend to be corrected; inertia implies that the correction will tend to overshoot, producing the need for a correction in the opposite direction and hence allowing for the possibility of propagating phenomena - acoustic pressure waves.

The tissue is characterized with some ambient parameters such as pressure $p_0$, density $\rho_0$ and velocity $\vec{v}_0$. There are three laws relating these parameters: (1) the conservation of mass; (2) the equation of motion of fluid and (3) the pressure-density relations. Based on these relations the wave equation of sound in tissue is derived. The following derivation of the wave equation is based on [21].
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Figure 3.1: On the left: non-moving volume $V$ with moving fluid. The time rate of mass within $V$ is equal to the mass flowing through the surface $S$. On the right: the mass leaving through the element area $\Delta S$ in time $\Delta t$ is equal to the mass in the slanted cylinder of length $|\vec{v}|\Delta t$, height $\vec{v} \cdot \Delta t$ and base area $\Delta S$

### 3.1.1 Conservation of mass

Consider Figure 3.1. For a fixed volume $V$ inside of a fluid such as air, water etc, the mass $m$ inside $V$ can be taken as the volume integral over of the density $\rho(\vec{x}, t)$, where $\vec{x}$ is a spatial point. The conservation of mass requires that the time rate of change of this mass to be equal to the net mass per unit time entering minus the net mass leaving that volume through the bounding surface $S$. The mass leaving the volume through a surface area $\Delta S$ is:

$$\Delta m = \rho(\vec{x}_s, t) \vec{v}(\vec{x}_s, t) \cdot \vec{n}(\vec{x}_s) \Delta S,$$  \hfill (3.1)

where $\vec{x}_s$ is a point on the surface $S$, $\vec{v}(\vec{x}_s, t)$ is the fluid velocity and the subscript $s$ means that $\vec{x}$ lies on the surface.

The mass leaving the volume per unit time is the surface integral over the surface $S$ of $\rho \vec{v} \cdot \vec{n}$, and from the law of conservation one gets:

$$\frac{\partial}{\partial t} \iiint_V \rho \, d\vec{x} = - \iint_S \rho \vec{v} \cdot \vec{n} \, d\vec{x}_S$$ \hfill (3.2)

After applying the Gauss theorem and some mathematical manipulations, the differential equation for for conservation of mass in fluid is obtained:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$ \hfill (3.3)

### 3.1.2 Euler’s equation of motion for a fluid

The second law of Newton states:

The acceleration of an object as produced by a net force is directly proportional to the magnitude of the net force, in the same direction as the net force, and inversely proportional to the mass of the object.

$$\frac{\partial \vec{v}}{\partial t} = \frac{\vec{f}}{m}$$ \hfill (3.4)
3.1. The wave equation

Moving surface

Origin
⃗n
∆S
V ∗(t)
⃗fB∆V ⃗xS
⃗v∆t
⃗fS∆S

Figure 3.2: Forces acting on a particle occupying volume $V^*$. Every particle located on the surface at coordinates $\vec{x}_S$ moves with velocity $\vec{v}(\vec{x}_S,t)$. The acting forces are the surface force per unit area $\vec{f}_S$ and the body force per unit volume $\vec{f}_B$.

This law was applied by Euler for fluids. Consider Figure 3.2. A fluid particle consists of the fluid inside some moving volume $V^*(t)$. Each point of the surface of the volume is moving with a local fluid velocity $\vec{v}(\vec{x}_S,t)$. The Newton’s law for the fluid particle can be expressed as:

$$\frac{d}{dt} \iiint_{V^*} \rho \vec{v} \, d\vec{x} = \iint_{S^*} \vec{f}_S d\vec{x}_S + \iiint_{V^*} \vec{f}_B \, d\vec{x}$$  \hspace{1cm} (3.5)

Here $\vec{f}_S$ is the apparent surface force per unit area, and $\vec{f}_B$ is the body force, e.g. that due to the gravity. The body force is assumed to have a negligible influence. If an ideal fluid with no viscosity is assumed, then the surface force is directed normally into the surface $S^*$ and is given by:

$$\vec{f}_S = -n \vec{p},$$  \hspace{1cm} (3.6)

where $p$ is the pressure\(^1\). The negative sign in the equation follows from the third law of Newton.

3.1.3 Pressure-density relations

A general formula for the relation between the pressure and density can be expressed as:

$$p = p(\rho)$$  \hspace{1cm} (3.7)

Many assumptions have been made, but it was Laplace who applied the simple principle that sound propagation occurs with negligible internal heat flow. For a gas, with constant heat coefficients, and for which the pressure is proportional to the density, this principle leads to the relation:

$$p = K\rho^\gamma$$  \hspace{1cm} (3.8)

3.1.4 Equations of linear acoustics

As previously stated the sound waves represent a propagation of a perturbation of the ambient state of the field. Usually their magnitude is significantly smaller compared to the ambient

\(^1\vec{f}_S\) is defined per unit area.
values characterizing the medium \((p_0, \rho_0, \vec{v}_0)\). The ambient variables satisfy the fluid-dynamic equations, but in the presence of disturbance one has:

\[
\begin{align*}
  p &= p_0 + p' \\
  \rho &= \rho_0 + \rho' \\
  \vec{v} &= \vec{v}_0 + \vec{v}'
\end{align*}
\] (3.9)

where \(p'\) and \(\rho'\) represent the acoustic contributions to the overall pressure and density fields.

The following assumptions are made:

- The medium is homogeneous. The values of the variables describing the medium are independent of their position.
- The medium is stationary. The properties of the medium are independent of time.
- The ambient velocity \(\vec{v}_0\) is zero. There is no flow of matter.
- The process is adiabatic. There is no heat flow in the medium.

Having the above assumptions, the linear approximation of the laws governing the propagation of waves in a fluid can be obtained:

\[
\begin{align*}
  \text{Conservation of mass} & \quad \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v'} = 0 \\
  \text{Fluid motion} & \quad \rho_0 \frac{\partial \vec{v'}}{\partial t} = -\nabla p' \\
  \text{Adiabatic process} & \quad p' = c^2 \rho'
\end{align*}
\] (3.10)

In the above equations \(c\) is the speed of sound in medium for longitudinal waves.

Further in the thesis the primes will be omitted for notational simplicity.

### 3.1.5 The wave equation

The propagation of sound is subject to the equations presented in the previous section. Consider the equation of preservation of mass:

\[
\frac{\partial p}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0
\]

By substituting \(p = \rho/c^2\) from the pressure-density relations one gets:

\[
\frac{1}{c^2} \frac{\partial p}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0
\]

Differentiating both sides of the equation with respect to time gives:

\[
\frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{\partial p}{\partial t} + \rho_0 \nabla \cdot \vec{v} \right) = 0
\]

\[
\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial \vec{v}}{\partial t} = 0
\]
Substituting the equation of fluid motion in the above result gives the wave equation for pressure:
\[ \nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \] (3.11)

The equation can be also expressed for the other two variables: \( \rho \) and \( \vec{v} \). If these are to be calculated, then two additional equations must be solved. To simplify the calculations a new variable, the velocity potential \( \Phi(\vec{x},t) \), is introduced. It is not a physical variable, but it can lead to an alternative formulation of the wave equation. In order to use the velocity potential an assumption that the acoustic field has no curl must be made:
\[ \nabla \times \vec{v} = 0 \] (3.12)

Usually this condition is satisfied by the sound fields. The velocity potential is defined by:
\[ \vec{v}(\vec{x},t) = \nabla \Phi(\vec{x},t) \] (3.13)

The relation between velocity potential \( \Phi \) and the pressure \( p \) is given by:
\[ p(\vec{x},t) = -\rho_0 \frac{\partial \Phi(\vec{x},t)}{\partial t} \] (3.14)

The wave equation (3.11) becomes:
\[ \nabla^2 \Phi(\vec{x},t) - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0 \] (3.15)

### 3.2 Solutions to the wave equation

Two classical methods for solving the wave equation exist, those of d’Lambert and Bernoulli. The former expresses the solution as a sum of two wave fields - one converging and one diverging. The latter assumes that the equation can be written as a product of functions that depend only on one variable.

In the following we will assume that the source is a delta function in space oscillating at a constant angular frequency \( \omega = 2\pi f \).

#### 3.2.1 Solution of the wave equation in Cartesian coordinates

According to Bernoulli’s method the function \( \Phi(\vec{x},t) = \Phi(x,y,z,t) \) must be assumed as a product of 4 functions:
\[ \Phi(x,y,z,t) = f(x)g(y)h(z)p(t) \] (3.16)

For simplicity let’s initially assume (see Section 2.2 in [22]) that \( \Phi(x,y,z,t) \) has a complex exponential form:
\[ \Phi(x,y,z,t) = \tilde{F} \exp \left( j\omega t - (k_x x + k_y y + k_z z) \right), \] (3.17)
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Figure 3.3: A snapshot of pressure distribution of a 2D plane wave. The gray level is proportional to the pressure magnitude.

where \( \hat{F} \) is a complex constant and \( k_x, k_y, k_z \) and \( \omega \) are real constants with \( \omega \geq 0 \). Substituting this form in the wave equation one gets:

\[
k_x^2 \Phi(x,y,z,t) + k_y^2 \Phi(x,y,z,t) + k_z^2 \Phi(x,y,z,t) = \frac{\omega^2}{c^2} \Phi(x,y,z,t)
\]

\[
\downarrow
\]

\[
k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}
\]

(3.19)

As long as the above constraint is satisfied, signals with the form \( \hat{F} \exp\{j[\omega t - (k_x x + k_y y + k_z z)]\} \) satisfy the wave equation (3.15).

This solution is known as a monochromatic plane wave and for a given point in space with coordinates \((x_0,y_0,z_0)\) it becomes:

\[
\Phi(x_0,y_0,z_0,t) = \hat{F} \exp \left( j \left[ \frac{\omega}{\text{frequency}} t - (k_x x_0 + k_y y_0 + k_z z_0) \right] \right)
\]

(3.20)

The observable signal at \((x_0,y_0,z_0)\) is a complex exponential with frequency \( \omega \). The wave is a plane wave because the phase is the same at the points lying on the plane given by \( k_x x + k_y y + k_z z = C \), where \( C \) is a constant. Figure 3.3 shows a snapshot of a plane wave. The time is frozen \((t = \text{const})\) and a 2D cross-section \((z = \text{const})\) is shown. The gray level in the image is proportional to the pressure. The wave propagates in a plane perpendicular to the \( z \) axis at an angle of 45\(^\circ\) to the \( x \) axis \((k_z = 0, k_x = k_y = k \cos \frac{\pi}{4})\). The distance between two points that have the same phase is called wavelength and is equal to:

\[
\lambda = \frac{c}{f}.
\]

(3.21)

The wave number and the wavelength are related through the equation:

\[
k = \frac{2\pi}{\lambda}
\]

(3.22)

If the pressure is taken along one of the axis, say \( y \), one will get a one-dimensional pressure distribution with wavelength \( \lambda_y = k_y/(2\pi) \).


3.2. Solutions to the wave equation

3.2.2 Solution of the wave equation in polar coordinates

The wave equation in spherical coordinates can be written as [22]:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial \Phi}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 \Phi}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}.
\] (3.23)

where \( \Phi \) is the velocity potential which is a function of the distance to the origin of the coordinate system \( r \) and the azimuth and elevation angles \( \theta \) and \( \phi \), and the time, \( \Phi = \Phi(r, \theta, \phi, t) \).

The general solutions are rather complicated. A solution of particular interest is when \( \Phi \) is independent of the angles \( \theta \) and \( \phi \):

\[
\frac{\partial \Phi}{\partial \theta} = \frac{\partial \Phi}{\partial \phi} = 0.
\] (3.24)

The wave equation reduces to:

\[
\frac{\partial^2}{\partial r^2} (r \Phi) = \frac{1}{c^2} \frac{\partial}{\partial t^2} (r \Phi).
\] (3.25)

Solutions to this equation are the spherical waves, which in complex form are given by:

\[
\Phi(r, t) = \frac{\hat{F}}{r} \exp(j(\omega t - kr)),
\] (3.26)

where \( \hat{F} \) is a complex amplitude, \( r \) is the distance to traveled by the wave and \( k \) is the wave number:

\[
k^2 = \frac{\omega^2}{c^2}.
\] (3.27)

The waves can be either diverging (going away from the source) or converging. The diverging wave can be expressed by:

\[
\Phi(r, t) = \frac{\hat{F}}{r} \exp(j \omega (t - \frac{r}{c})).
\] (3.28)

Figure 3.5 shows a 2D snapshot of the pressure distribution. The amplitude of the pressure is normalized. The figure is not exact, since at the source of the spherical wave the radius \( r \) tends to zero (\( r \to 0 \)), and the function should go to infinity.
3.3 Theory of radiation and diffraction

The wave equation describes the propagation of waves in the free space. To describe the radiation of ultrasound the source and propagation are linked together through boundary conditions. There are two classical solutions to the problem: the Kirchoff and the Rayleigh-Sommerfeld theories of diffraction [23]. The simulation program Field II used in this thesis is based on a method derived from the Rayleigh-Sommerfeld theory and this will be examined in the next section. Then a brief description of the numerical calculation of the pressure field through the means of spatial impulse responses will be given. Finally some approximation to the solutions will be presented.

3.3.1 Rayleigh integral

Figure 3.6 shows the basic setup of the problem. The aperture is planar and lies on an infinite rigid baffle, on which the velocity to the plane is zeros except for the aperture. The problem is to find the field in a point with spatial coordinates $\vec{x}_1$. The coordinates of a point lying on the aperture is denoted with $\vec{x}_0$. The velocity potential at the point is given by the Rayleigh integral [21, 23]:

$$\Phi(\vec{x}_1, t) = \iint_\Sigma \frac{v_n(\vec{x}_0, t - \frac{|\vec{x}_1 - \vec{x}_0|}{c})}{2\pi|x_1 - x_0|} d\vec{x}_0, \quad (3.29)$$

where $\Sigma$ is the area of the aperture. The Rayleigh integral is basically a statement of Huygens’ principle that the field is found by summing the contributions from all infinitely small area elements that make up the aperture. The pressure is given by:

$$p(\vec{x}_1, t) = \frac{\rho_0}{2\pi} \iint_\Sigma \frac{\partial v_n(\vec{x}_0, t - \frac{|\vec{x}_1 - \vec{x}_0|}{c})}{|\vec{x}_1 - \vec{x}_0|} d\vec{x}_0 \quad (3.30)$$

This solution is given for a rigid baffle. There is a solution for a soft baffle which requires that the pressure on the baffle is equal to zero. The solution is the same except for an obliquity term:
3.3. Theory of radiation and diffraction

\[ \Phi(\vec{x}_1, t) = \int \int_{\Sigma} \frac{v_n(\vec{x}_0, t - \frac{|\vec{x}_1 - \vec{x}_0|}{c})}{2\pi|\vec{x}_1 - \vec{x}_0|} \cdot \cos \theta \, d\vec{x}_0, \]  

(3.31)

where \( \theta \) is the angle between the vector \( \vec{x}_1 \) and the normal vector to the aperture plane. The pressure is given by:

\[ p(\vec{x}_1, t) = \frac{p_0}{2\pi} \int \int_{\Sigma} \frac{\partial}{\partial t} v_n(\vec{x}_0, t - \frac{|\vec{x}_1 - \vec{x}_0|}{c})}{|\vec{x}_1 - \vec{x}_0|} \cdot \cos \theta \, d\vec{x}_0 \]  

(3.32)

Usually the true value of the field is found between the values of the two solutions.

3.3.2 Spatial impulse responses

The spatial impulse response method applies a linear system theory to the wave equation and it separates the temporal and spatial characteristics of the acoustical field, which is apprehended as a spatial filter. The following considerations are based on [24]. The assumptions are that the medium is non-attenuating and homogeneous, and that the normal component \( v_n \) of the velocity \( \vec{v} \) is uniform across the aperture. The excitation can be separated from the transducer geometry by using a convolution:

\[ \Phi(\vec{x}_1, t) = \int \int_{\Sigma} v_n(\vec{x}_0, t) \delta \left( \vec{x}_0, t - \frac{|\vec{x}_1 - \vec{x}_0|}{c} \right) \frac{1}{2\pi|\vec{x}_1 - \vec{x}_0|} \, dt \, d\vec{x}_0, \]  

(3.33)

where \( \delta \) is the Dirac delta function. Since the velocity is uniform across the aperture the integral can be written out as:

\[ \Phi(\vec{x}_1, t) = v_n(t) \int \int_{\Sigma} \delta \left( t - \frac{|\vec{x}_1 - \vec{x}_0|}{c} \right) \frac{1}{2\pi|\vec{x}_1 - \vec{x}_0|} \, d\vec{x}_0, \]  

(3.34)

where * denotes convolution in time. The integral in the equation:

\[ h(\vec{x}_1, t) = \int \int_{\Sigma} \delta \left( t - \frac{|\vec{x}_1 - \vec{x}_0|}{c} \right) \frac{1}{2\pi|\vec{x}_1 - \vec{x}_0|} \, d\vec{x}_0 \]  

(3.35)
is called the spatial impulse response because it varies with the relative position of the transducer and the point in space. The pressure as a function of time at one given point in space can be found by:

\[ p(\vec{x}_1, t) = \rho_0 \frac{\partial v_n}{\partial t} \ast h(\vec{x}_1, t). \]  
\[ \text{(3.36)} \]

The use of spatial impulse response is a powerful technique and analytical solutions for several geometries exist such as concave transducers [25], triangle apertures [26, 27], or apertures that can be defined as planar polygons [28].

### 3.3.3 Fresnel and Fraunhofer approximations

Simplification to the Rayleigh integral can be made by using some geometrical considerations. The derivations are carried out for the continuous wave (CW) case. The Rayleigh integral for the CW case can be found from (3.34) using a Fourier transform and the following relations:

\[ h(t) \ast g(t) \Leftrightarrow H(f) \cdot G(f) \]  
\[ \delta(t - T) \Leftrightarrow j\omega \exp(-j\omega T) \]  
\[ \text{(3.37, 3.38)} \]

Further an assumption will be made that the velocity distribution over the aperture can be defined as a separable function:

\[ v_n(\vec{x}_0, t) = v_n(t) a(\vec{x}_0), \]  
\[ \text{(3.39)} \]

where \( a(\vec{x}_0) \) is a weighting function of the contribution of the individual points, known also as “apodization function”. The apodization function becomes zero outside the aperture limits.

For a single frequency the temporal function \( v_n(t) \) will be assumed to be:

\[ v_n(t) = v_0 \exp(j\omega t). \]  
\[ \text{(3.40)} \]

For a single frequency the Rayleigh integral becomes:

\[ \Phi(\vec{x}_1, \omega) = v_0 \exp(j\omega) \frac{jk}{2\pi} \int_{-\infty}^{\infty} a(x_0, y_0) \frac{\exp(-jkr_0)}{r_0} \, dx_0 \, dy_0 \]  
\[ \text{(3.41)} \]
where \( r_{01} \) is the distance between the field point and a point in aperture. Figure 3.7 illustrates the geometry of the radiation problem. The aperture lies on a plane parallel to the \((x, y)\) axes of the coordinate system and a depth \( z = 0 \). The coordinates in this plane are denoted with sub-script \( 0 \). The radiated field is sought on a plane parallel to the emitting aperture. The distance between the two planes is equal to \( z_1 \). The coordinates of the points in the plane where the radiated field is calculated have a sub-script \( 1 \). So \( \vec{x}_0 = (x_0, y_0, 0) \) is a source point, and \( \vec{x}_1 = (x_1, y_1, z_1 = \text{const}) \) is a field point. The distance between the two points is:

\[
    r_{01} = |\vec{x}_1 - \vec{x}_0| = \sqrt{z_1^2 + (x_1 - x_0)^2 + (y_1 - y_0)^2}.
\]

(3.42)

The Fresnel approximation assumes that the changes in the amplitude are not as strongly expressed as the changes in the phase of the field [23]. The distance in the denominator is then approximated:

\[
    r_{01} \approx z_1.
\]

(3.43)

The second approximation is done by extending the term \( r_{01} \) which is part of the exponent into Taylor series:

\[
    r_{01} \approx z_1 \left( 1 + \frac{1}{2} \left( \frac{x_1 - x_0}{z_1} \right)^2 + \frac{1}{2} \left( \frac{y_1 - y_0}{z_1} \right)^2 \right).
\]

(3.44)

In this way the spherical waves are substituted with parabolic waves. The velocity potential at a point in the plane \((x_1, y_1, z_1 = \text{const})\) becomes

\[
    \Phi(x_1, y_1, z_1, \omega) = v_0 \exp(\text{j} \omega t) \frac{jk \exp(-jkz_1)}{2\pi z_1} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -\frac{jk}{2z_1} ((x_1 - x_0)^2 + (y_1 - y_0)^2) \right) \, dx_0 \, dy_0,
\]

(3.45)

This result is known as the Fresnel diffraction integral. When this approximation is valid the observer is said to be in the region of Fresnel diffraction, or equivalently, in the \textit{near field} of the aperture. This is the usual working region for the ultrasound scanners.

Rearranging the terms in (3.45) one gets:

\[
    \Phi(x_1, y_1, z_1, \omega) = v_0 \exp(\text{j} \omega t) \frac{jk \exp(-jkz_1)}{2\pi z_1} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(x_0, y_0) \exp \left( -\frac{jk}{2z_1} ((x_1 - x_0)^2 + (y_1 - y_0)^2) \right) \, dx_0 \, dy_0 \quad (3.46)
\]

\[
    h'(x_0, y_0; x_1, y_1) = \exp \left( -\frac{jkx_1^2 + ky_1^2}{2z_1} \right) \exp \left( -\frac{jkx_0^2 + ky_0^2}{2z_1} \right) \exp \left( \frac{jkx_1x_0 + ky_0y_1}{z_1} \right) \quad (3.48)
\]

The Fraunhofer assumption is that the distance between the aperture plane and the observation plane is much greater than the dimensions of the aperture function, i.e.:

\[
    z \gg \frac{k \max(x_0^2 + y_0^2)}{2}.
\]

(3.47)

By developing the term \((x_1 - x_0)^2 + (y_1 - y_0)^2\), the exponential in the integral becomes:

\[
    h'(x_0, y_0; x_1, y_1) = \exp \left( -\frac{jkx_1^2 + ky_1^2}{2z_1} \right) \exp \left( -\frac{jkx_0^2 + ky_0^2}{2z_1} \right) \exp \left( \frac{jkx_1x_0 + ky_0y_1}{z_1} \right) \quad (3.48)
\]
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Figure 3.8: Idealized continuous pressure field for flat, round transducer (aperture).

The Fraunhofer approximation becomes (skipping the temporal terms):

\[
\Phi(x_1, y_1, z_1) = \frac{jk \exp(-jkz_1) \exp \left(-j \frac{k}{z_1} (x_1^2 + y_1^2)\right)}{2\pi z_1} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(x_0, y_0) \exp \left(jk \left( \frac{x_1 x_0}{z_1} + \frac{y_1 y_0}{z_1} \right) \right) dx_0 \, dy_0
\]

(3.49)

Aside from the quadratic phase term outside the integral, this result is the Fourier transform\(^2\) of the apodization function \(a(x_0, y_0)\) at frequencies:

\[
f_x = \frac{x_1}{\lambda z_1}
\]
\[
f_y = \frac{y_1}{\lambda z_1}
\]

(3.50)

The region in which the Fraunhofer approximation is valid is known as the far field. The depth \(z_1\) usually used as the border (see Figure 3.8) of the transition from near to far field is given by the following:

\[
z_1 = \frac{r^2}{\lambda},
\]

(3.51)

where \(2r\) is the lateral size of the aperture. In the near field the main part of the beam is confined to lie within the extent of the transducer surface. In the far field the beam starts to diverge. The angle of divergence is:

\[
\theta_a = \arcsin \left(0.61 \frac{\lambda}{r}\right).
\]

(3.52)

The Fraunhofer approximation gives a very nice Fourier relation between the apodization function and the generated field. A similar Fourier relation between the aperture apodization function and the radiated field exists also for the Fresnel zone at the focal plane. The derivation is given in Appendix A. Sometimes it is said that the focusing brings the far field conditions in the near field.

3.4 Propagation in tissue

The methods concerned in this thesis are the pulse-echo methods, meaning that a pressure wave is transmitted into the region under investigation and the received echo is displayed. The

\(^2\)The sign in the exponent term can be either plus or minus depending on the chosen positive direction of the the phase. For a discussion on this subject see Introduction to Fourier optics [23].
received echo is based on two distinct phenomena: reflection and scattering.

### 3.4.1 Reflection

Reflection occurs at the border of two regions with different acoustic impedances. For a monochromatic wave the relation between particle velocity \( u \) and the pressure \( p \) is given through the acoustic impedance \( Z \) [21]:

\[
    u = \frac{p}{Z}.
\]

For a plane progressing wave the impedance is:

\[
    Z = \rho_0 c.
\]

The situation of reflection and transmission of a plane wave propagating obliquely to the reflecting surface is shown in Figure 3.9. If the first medium has a speed of sound \( c_1 \) and second medium \( c_2 \), then the angles of transmitted and reflected waves are described by Snell’s law:

\[
    \frac{c_1}{c_2} = \frac{\sin \theta_i}{\sin \theta_r},
\]

where \( \theta_i \) and \( \theta_r \) are the angles of the incident and transmitted waves, respectively. If the angles are measured with respect to the normal vector to reflecting surface then:

\[
    \theta_r = -\theta_i.
\]

The pressure amplitude transmission coefficient is given by:

\[
    \frac{p_t}{p_i} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_r},
\]

where \( Z_1 \) and \( Z_2 \) are the acoustic impedances of the first and the second media, respectively, and \( p_i \) and \( p_t \) are the respective amplitudes of the pressure.
3.4.2 Scattering

A single scatterer and the point spread function.

The spatial pulse-echo response for a single scatterer corresponds to the convolution of the spatial impulse responses of the transmit and receive apertures [29]. This can be easily justified since a point can be represented as a delta function $\delta(x_p, y_p, z_p)$ and the system is assumed to be linear. In the rest of the dissertation the pulse-echo response of a point will be called point spread function. The point spread function can be in space $P(x_p, y_p, z_p)$, or in space and time. In the latter case this usually represents a collection of the responses along a line of points.

As discussed in Section 3.3, there is a Fourier relation between the apodization of a focused transducer and the point spread function in the focus. The point spread functions must be defined on a spherical surface, so often they will be given as a function of angle.

Multiple scatterers

Figure 3.10 shows the measurement situation. A transducer insonifies a group of scatterers (the operation is shown with a thick gray arrow). The wave reaches the scatterers and they start to vibrate, becoming omnidirectional sources of spherical waves (Rayleigh scattering). The scattering is assumed to be weak, i.e. a wave generated by a scatterer cannot be scattered again by another scatterer (Born approximation [21]). The medium is assumed to be linear and homogeneous, and the total response of the field is the sum of the responses of the individual scatterers. In the thesis yet another approximation will be used - the one of separability of the excitation from the transducer geometry. The received response becomes [30]:

$$p_r(\vec{x}, t) = v_{pe}(t) * f_m(\vec{x}) * h_{pe}(\vec{x}, t),$$  \hspace{1cm} (3.58)
3.5. Beamforming

where $v_{pe}$ includes the transducer excitation and the impulse responses of the transmitting and receiving apertures, $f_m$ is the scatterer map, and $h_{pe}$ is the pulse echo spatial response. The scatterer function $f_m$ is a function of the two phenomena causing the scattering - the change in density and speed of sound:

$$f_m(x) = \frac{\Delta \rho(x)}{\rho_0} - \frac{2\Delta c(x)}{c_0}.$$  \hfill (3.59)

**3.4.3 Attenuation**

The human tissue is a lossy medium and the ultrasound pulse is attenuated as it propagates in it. This attenuation is not described by the linear wave equation. Most of the energy loss is due to absorption. The general solution is rather complicated, but a rather simple approximation can be made expressing the losses in dB/(MHz·cm) [31].

**3.5 Beamforming**

This section provides a description of the beamforming used in the modern scanners. The type of beamforming used in them is known also as a time domain beamforming. The frequency domain beamforming methods [22, 32, 33] are neither spread in the medical ultrasound scanners, nor used in the synthetic aperture algorithms presented in the rest of the thesis, and will not be described here.

Figure 3.11 shows a typical delay and sum beamformer [22, 34, 35, 36, 37]. The depicted array is a linear array of piezoelectric transducer elements. It transmits a sound pulse into the body and receives echoes from the scattering structures within. The transmit and receive signals can be individually delayed in time, hence the term phased array. Through the time delays the beam can be steered in a given direction and focused at a given axial distance both in transmit and receive. Figure 3.11 shows focusing during reception. Using simple geometric relations the transducer can be focused at any point. There is a Fourier relation between the aperture weighting function and the point spread function at the focal depth. The point spread function
determines the imaging capabilities of the system. It consists of a main lobe and side lobes. The narrower the main lobe, the higher the resolution of the system, hence larger arrays are desirable. The side lobes are caused by the finite size of the aperture (the edges). Applying weighting on the aperture function has the same effect as applying weighting on the Fourier transform of a signal - the side lobes decrease. The beamformation procedure becomes (see: Figure 3.11):

\[
s(t) = \sum_{i=1}^{N_{\text{edc}}} a_i r_i(t - \tau_i),
\]

(3.60)

where \(N_{\text{edc}}\) is the number of transducer elements, \(a_i\) are the apodization coefficients and \(\tau_i\) are the applied delays. Usually in transmit the focus is fixed. In receive, however, the focus can be changed as a function of time thus “tracking” the current position of the wave front. This is usually denoted as dynamic focusing. The modern digital scanners can change the delays for every sample in the beamformed scan lines \(s(t)\). There are different ways to define the scan geometry. The approach adopted in this work and used in the Beamformation Toolbox [38], and in Field II [19] is to define the scan lines using a “focus center” and a “focus point” or a direction in the case of dynamic focusing.

Figure 3.12 shows a 2D geometry for determining the delays \(\tau\). The equations will, however, be done for 3D geometry. The assumption is that a plane wave confined in space is transmitted from the center \(\vec{x}_c = (x_c, y_c, z_c)\) and propagates along the line defined by the center point and the focal point \(\vec{x}_f = (x_f, y_f, z_f)\). The transducer element \(i\) with coordinates \(\vec{x}_i = (x_i, y_i, z_i)\) transmits sound pulse. The pulse from the transducer element must arrive at the same time with the imaginary pulse transmitted by the center point.

The delay with respect to the trigger is given by:

\[
\tau_i = \frac{|\vec{x}_c - \vec{x}_f| - |\vec{x}_i - \vec{x}_f|}{c}.
\]

(3.61)

Usually the origin of the coordinate system lies in the geometric center of the transducer array. It is possible to have multiple focusing points (focal zones) along the beam. In this case it is necessary to define the starting times at which the next delay profiles will be active. Usually the time is set as the time of flight of ultrasound pulse from \(\vec{x}_c\) to \(\vec{x}_f\). The fully dynamic focusing implies that the delay profile is recalculated for every point (sample) along the scan line. In this
3.6. Resolution

Figure 3.13: The system is characterized by axial, lateral and elevation resolution.

case the scan lines are defined by a center (origin of the line) and a direction determined by the azimuth and elevation angles.

In digital systems the received signal $r(t)$ is sampled at a sampling frequency $f_s$. The delays that can exactly be generated are multiples of the sampling period $T_s = 1/f_s$. If a delay is needed which is a fraction of $T_s$, then some interpolation is performed. The nearest neighbor interpolation is the simplest, but is not adequate in most cases [39]. The linear interpolation meets most of the practical needs of B-mode imaging, and is the one implemented in the experimental system RASMUS [40, 41, 42], which was developed at CFU. This is the type of interpolation used in the Beamformation Toolbox when calculation speed is necessary [43]. For some applications the linear interpolation is not adequate. One such application is the motion compensation scheme presented in Chapter 12. There the interpolation is done by splitting the delays into two parts - a coarse delay and a fine delay. The coarse delay is a multiple of the sampling period and is achieved by shifting the signal with a number of samples. The fine delay is implemented by filtering the shifted signal with a FIR fractional delay filter [44]. The filters form a filter bank, whose size depends on how precise the fractional delay should be. It is also possible to achieve fractional delays with theoretically unlimited precision if the filter coefficient is calculated at the moment of beamformation (see http://www-ccrma.stanford.edu/~jos/resample/). The Beamformation Toolbox provides means for creating a filter bank.

As discussed previously, the focus in transmit is fixed. There exist approaches to compensate for it [45, 46], but they are not what is meant to be “conventional” focusing.

3.6 Resolution

Figure 3.13 shows the three dimensional nature of the resolution cell of an ultrasound system. The axial resolution is inversely proportional to the bandwidth of the system[47]. If the transducer is not 1.5, 1.75 or 2D matrix array [48, 11], the resolution in the elevation plane is determined by the geometry of the transducer. The resolution in the azimuth plane is the only one dependent on the focusing algorithms and is usually the one considered in the thesis. For conventional B-mode scans, the information is considered as gathered only from the $(x-z)$ plane. There are many ways to determine the azimuth resolution [22]. In this thesis the resolution will be stated as the beam width at a given level, most often at -6 dB.
Figure 3.14: (Left) An aperture of size $D$ composed of elements with size $w$ whose centers are positioned at distance $d_x$ apart is modeled as $\Pi(x_0/D)\Pi(x_0/w)\ast\text{III}(x/d_x)$. The radiation pattern (right) is $\text{sinc}(x_1/(\lambda z_1)w) \cdot \left[ \text{sinc}(x_1/(\lambda z_1)D) \ast \text{III}(x_1/(\lambda z_1)d_x) \right]$. 

The systems are assumed to be linear, and therefore they can be characterized fully by the point spread function. Usually this is the point spread function along the azimuth direction (along $x$) for linear scans, and along an arc for sector scans. The plots shown will be obtained from the projection of the maximum of the point spread function:

$$A_{t/r}(x) = \max_z p_{t/r}(x,z). \quad (3.62)$$

For the 3D scans, the point spread function will be the projection along the axial direction on a 2D surface. Usually what matters are the relative levels of the point spread function at a given spatial position relative to the level at the focal point. Hence the plots will be normalized, and most often logarithmically compressed to emphasize the details.

### 3.7 Frequency domain representation of the system

As seen from Section 3.3.2, the linear systems approach can be used in ultrasound systems to calculate the pulse echo response of the system. Frequency domain methods are a powerful tool in engineering, and it turns out that they can be also used for ultrasound scanners.

In Section 3.3.3 and in Appendix A it is shown, that there is a Fourier relation between the apodization function of the aperture and its radiation pattern in the focal plane (at small angles the spherical cap can be approximated with the tangent plane). If the scale factors in (A.9) are skipped, then the radiation patterns of the transmit and receive apertures can be expressed as:

$$A_t(x_1,y_1;z_f) = \mathcal{F}\{a_t(x_0,y_0)\} \quad (3.63)$$

$$A_r(x_1,y_1;z_f) = \mathcal{F}\{a_r(x_0,y_0)\} \quad (3.64)$$

where $A_t$ and $A_r$ are the radiation patterns of the transmit and receive aperture, respectively, and $a_t$ and $a_r$ their apodization functions. The focal point lies on a spherical cap at a distance $z_f$ away from the transducer. One of these relations is shown in Figure 3.14. The width of the array is $D$ and is modeled by the function $\Pi(x_0/D)$ where:

$$\Pi(x) = \begin{cases} 1 & -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (3.65)$$
The elements have width \( w \) and each of them is modeled by \( \Pi(x_0/w) \). The discrete nature of the array is modeled as a comb function \( \text{III}(x_0/d_x) \), which is a sum of offset delta functions:

\[
\text{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n). \quad (3.66)
\]

The apodization function of the array is then:

\[
a_r(x_0) = \text{III}(x_0/D)[\Pi(x_0/w) \ast \text{III}(x_0/d_x)] \quad (3.67)
\]

The radiation pattern of \( a_t(x) \) is found through the Fourier transform:

\[
\text{III}(x_0/D)[\Pi(x_0/w) \ast \text{III}(x_0/d_x)] \Leftrightarrow \text{sinc} \left( \frac{x_1}{k_{z_1}}w \right) \cdot \text{sinc} \left( \frac{x_1}{k_{z_1}}D \right) \ast \text{III} \left( \frac{x_1}{k_{z_1}}d_x \right) \] \quad (3.68)

where the scaling coefficients have been skipped for notational simplicity. From Figure 3.14 it can be seen that the radiation pattern consists of a main lobe, side and grating lobes. The width of the main lobe is inversely proportional to the width of the aperture.

The two-way radiation pattern is the product of the transmit and receive radiation patterns:

\[
A_{t/r}(x_1,y_1;z_f) = A_t(x_1,y_1;z_f) \cdot A_r(x_1,y_1;z_f). \quad (3.69)
\]

A fictional aperture that would have a radiation pattern equal to the two-way radiation pattern of the system will be called effective aperture [49, 50, 51]. Using the properties of the Fourier transform the apodization function of the effective aperture can be found by the spatial convolution of the apodization functions of the transmit and receive apertures:

\[
a_{t/r}(x_0,y_0) = a_t(x_0,y_0) \ast a_r(x_0,y_0). \quad (3.70)
\]

There is another term, ”co-array” [22, 52], which is by principle the same.

A linear system is characterized in frequency domain by its transfer function \( H(\omega) \) which is related to the impulse response of the system \( h(t) \) by the Fourier transform [53]:

\[
H(\omega) \equiv h(t). \quad (3.71)
\]

An ultrasound system can be characterized in a similar manner, by taking the Fourier transform of the pulse echo response. This characteristics varies in space and is useful mostly for the far field or for the focus. If the imaging system is considered only in the two-dimensional case, in the azimuth plane, then the 2D Fourier transform is involved. The result is given in spatial frequencies cycles/m or alternatively rad/m. In the latter case the frequencies are \( k_x \) and \( k_z \). They are related to the wave number \( k \) by:

\[
k_x^2 + k_z^2 = k^2. \quad (3.72)
\]

The bandwidth in the \( k_x \) and \( k_z \) domains determine the lateral and axial resolutions of the system, respectively. Because the letter \( k \) is used for notation this representation of the system is also known as the \( k \)-space [54, 55], or alternatively as wave number representation, angular spectrum [23], or plane wave decomposition [37]. The name angular spectrum comes from the fact that the spatial frequencies are related to the direction of propagation of a plane wave which is defined by an angle \( \theta \):

\[
k_x = k \sin \theta
\]

\[
k_z = k \cos \theta \quad (3.73)
\]
For the 3D case the azimuth angle also comes into play. The term “k-space” will be used in the rest of the thesis.

In Section 3.3.2 it was assumed that for the time function of the excitation can be separated from the transducer geometry. The Fourier transform of a separable function is also a separable function. The radiation pattern of an aperture is related to the apodization function through the Fourier transform. Taking the Fourier transform of the radiation pattern gives a function which is a scaled version of the aperture function. So the lateral bandwidth of the system $[\min(k_x), \max(k_x)]$ is determined by the spatial extent of the aperture. The larger the aperture, the bigger the bandwidth, the higher the resolution. The same applies for the axial resolution. Its bandwidth is determined by the excitation and the impulse response of the transmit and receive apertures. A higher bandwidth transducer gives a higher axial resolution. The ideal system is therefore an all pass filter.

The assumptions are that linear system theory can be applied for the ultrasound systems [24]. As so the k-space of the system can be increased by synthesizing a large effective aperture using synthetic aperture imaging algorithms which are the topic of the rest of this thesis.