Coded excitations

One of the major problems of all synthetic aperture imaging techniques is the signal-to-noise ratio. The signal level decreases not only due to the tissue attenuation but also because of the very nature of the emissions - sending a spherical wave. This problem can be addressed in several ways: (1) using multiple elements to create virtual sources, (2) generating long pulses with temporal encoding, and (3) transmitting from several virtual sources at the same time using spatial encoding.

The use of a long waveform is a well known and used technique in the radar systems. It involves either sending a long phase/frequency modulated pulse (PM or FM), or a series of short pulses with binary encoding.

Another way of encoding is the spatial encoding. Instead of sending only from a single virtual source, the transmission is done from all of the virtual sources at the same time and a binary encoding/decoding scheme is used to resolve the signals.

This chapter will start by discussing the properties of the matched filter, which is normally applied on the recorded data. The gain in signal-to-noise ratio and the obtained resolution will be given in terms of the signal duration and bandwidth. Then the pulse compression of linear frequency modulated pulses by using matched filter will be presented. The discussion on pulse compression will conclude with the binary Golay codes. Since the design of coded in time excitations goes beyond the scope of this work, the FM pulses used further on in the dissertation will be based on results obtained by my colleague Thanassis Misaridis.

Then the focus will be shifted towards the spatial encoding. The considerations will be limited to the use of Hadamard matrices for coding and decoding. The gain in signal-to-noise ratio will be briefly considered.

At the end of the chapter B-mode images from measurements on a phantom and in-vivo are presented. The scans are carried out with spatial, temporal and spatio-temporal encoding.

9.1 Temporal coding

9.1.1 The matched filter receiver

One of the steps in the reconstruction of images using synthetic aperture is the matched filtration (see Chapter 4). The matched filter also increases the signal-to-noise ratio which in an ultrasound system is limited by the peak acoustic power, rather than average power [115]. This
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condition can be expressed as:

\[
PSNR = \max \left( \frac{(r_m(t))^2}{P_N} \right),
\]

(9.1)

where \(r_m(t)\) is the output signal from the filter and \(P_N\) is the average noise power. The filter which maximizes this ratio is the matched filter [47]. Consider a passive filter with a transfer function \(H_m(f)\) The absolute magnitude of the signal after the filter can be expressed in terms of the complex spectrum of the input signal \(\hat{R}_i(f)\) as:

\[
|r_m(t)| = \left| \int_{-\infty}^{\infty} \hat{R}_i(f) \hat{H}_m(f) e^{j2\pi ft} \, df \right|.
\]

(9.2)

The noise power after the filter is given by:

\[
P_N = P_0 \int_{-\infty}^{\infty} |H_m(f)|^2 \, df,
\]

(9.3)

where \(P_0\) is the power density of the noise. By using Schwartz’s inequality:

\[
\left| \int_{-\infty}^{\infty} \hat{R}(f) \hat{H}(f) \, df \right|^2 \leq \int_{-\infty}^{\infty} |\hat{R}(f)|^2 \, df \int_{-\infty}^{\infty} |\hat{H}(f)|^2 \, df
\]

one gets:

\[
PSNR \leq \frac{\int_{-\infty}^{\infty} |\hat{R}_i(f)|^2 \, df \int_{-\infty}^{\infty} |H_m(f)|^2 \, df}{P_0 \int_{-\infty}^{\infty} |H_m(f)|^2 \, df},
\]

(9.4)

\[
|e^{j2\pi ft_1}| = 1
\]

(9.5)

\[
PSNR \leq \frac{\int_{-\infty}^{\infty} |\hat{R}_i(f)|^2 \, df}{P_0}.
\]

(9.6)

The equality is only obtained when:

\[
\hat{H}_m(f) = G_a \left[ \hat{R}_i(f) e^{j2\pi ft_1} \right]^* \]

(9.7)

\[
\hat{H}_m(f) = G_a \hat{R}_i^*(f) e^{-j2\pi ft_1}
\]

(9.8)

where \(t_1\) is a fixed delay, and \(G_a\) is a constant usually taken to be one. The impulse response of the filter in the time domain is then:

\[
h_m(t) = G_a r_i(t_1 - t).
\]

(9.9)

The filter that maximizes the SNR, thus has an impulse response equal to the input waveform with reversed time axis, except for an insignificant amplitude factor and translation in time. For this reason the filter is called matched filter. Aside from the amplitude and the phase constants,
9.1. Temporal coding

The transfer function of the matched filter is the conjugate of the signal spectrum. This means that maximization of the SNR is achieved by (1) removing any nonlinear phase function of the spectrum and (2) weighting the received spectrum in accordance with the strength of the spectral components in the transmitted signal.

Consider the one-dimensional measurement situation depicted on Figure 9.1. The pulse generator generates a radio frequency pulse \( g(t) \). The pulse is fed into a transducer with an electro-mechanical impulse response \( h(t) \). The generated pressure pulse \( p(t) = g(t) \ast h(t) \) is sent into the tissue. There the pulse propagates until it reaches a point scatterer which scatters the pulse back. The propagation of the pulse in the tissue can be expressed as a convolution between the pressure pulse and the spatial impulse response (see Section 3.3.2). For the simplified case, the spatial impulse response will be assumed a delta function, which is valid for the focus and for the far field. The back-scattered signal, which reaches the transducer is \( p(t) \ast \delta(t - t_1) \), where \( t_1 \) is the round-trip delay. The transducer converts the received pressure back to an electric signal \( r_i(t) \), which is:

\[
r_i(t) = g(t) \ast h(t) \ast h(t) \ast \delta(t - t_1).
\]

(9.10)

Assuming an ideal transducer with an impulse response \( h(t) = \delta(t) \) one gets that the signal received by a single scatterer is a replica of the transmitted pulse delayed in time:

\[
r_i(t) = g(t - t_1).
\]

(9.11)

The output \( r_m(t) \) of the matched filter with impulse response \( h_m(t) = r_i^*(t) \) is the autocorrelation function of the transmitted pulse:

\[
r_m(t) = R_{gg}(t),
\]

(9.12)

\[
R_{gg}(\tau) = \int_{-\infty}^{\infty} g(t)g^*(t + \tau) \, dt,
\]

(9.13)
where the delay $t_1$ has been skipped for notational simplicity. The latter means that the maximum will occur at a lag corresponding to the two-way propagation time. The displayed image is the envelope of the autocorrelation function of the transmitted pulse. It can be seen that the imaging abilities of the system depend on the parameters of the autocorrelation function and therefore on the parameters of the transmitted signal.

### 9.1.2 Signal parameters

The signals used in ultrasound are usually RF pulses of the form:

$$ g(t) = w(t) \cos[2\pi f_0 t + \phi(t)], \quad (9.14) $$

where $w(t)$ is a baseband modulation signal, $f_0$ is the carrier frequency, and $\phi(t)$ is the phase of the signal. For the sake of simplicity in the analysis the complex notation for the signals will be used:

$$ \dot{g}(t) = w(t)e^{j\phi(t)} \quad (9.15) $$

$$ \dot{\phi}(t) = w(t) $$

where $w(t)$ and $\phi(t) = 2\pi f_0 t + \phi(t)$ are the amplitude and the phase, respectively.

Consider Figure 9.2 showing two RF pulses at a center frequency $f_0 = 5$ MHz. It can be seen that these pulses are localized in time, and therefore their power density spectra are infinite. In spite of the infinite nature of the spectra, most of the energy is concentrated around the central frequency $f_0$ within some band $\Delta f$ which can be given in terms of the standard deviation [47, 116], and will be called RMS bandwidth. The energy contained in the signal is:

$$ ||\dot{g}(t)||^2 = \int_{-\infty}^{\infty} |\dot{g}(t)|^2 \, dt = \int_{-\infty}^{\infty} |\dot{G}(f)|^2 \, df = E_s \quad (9.17) $$

Figure 9.2: Examples of two radio pulses with center frequency $f_0 = 5$ MHz. The left column shows a pulse with a Gaussian window applied on it, and the right with a rectangular window applied on. The top row shows the signals in time domain, and bottom row in frequency domain. The windowing functions are drawn in red.
The mean time, and mean frequency can be found by:

\[ \bar{t} = \frac{1}{E_s} \int_{-\infty}^{\infty} t |\dot{g}(t)|^2 \, dt \]  \hfill (9.18)

\[ \bar{f} = \frac{1}{E_s} \int_{-\infty}^{\infty} f |\dot{G}(f)|^2 \, df. \]  \hfill (9.19)

Following the consideration in Section 2.4 in [116], the mean frequency \( \bar{f} \) is found to be:

\[ \bar{f} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\phi(t)}{dt} \frac{|\dot{g}(t)|^2}{E_s} \, dt, \]  \hfill (9.20)

which says that the mean frequency \( \bar{f} \) is the weighted average of the instantaneous quantities \( \frac{d\phi(t)}{dt} \) over the entire time domain. The value \( \frac{d\phi(t)}{dt} = f_i(t) \) is named the mean instantaneous frequency and is:

\[ f_i(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} = f_0 + \frac{1}{2\pi} \frac{d\phi(t)}{dt} \]  \hfill (9.21)

The RMS time duration \( 2\Delta t \) and the RMS frequency bandwidth \( 2\Delta f \) are defined as:

\[ \Delta t^2 = \int_{-\infty}^{\infty} (t - \bar{t})^2 \frac{|\dot{g}(t)|^2}{E_s} \, dt \]  \hfill (9.22)

\[ \Delta f = \int_{-\infty}^{\infty} (f - \bar{f})^2 \frac{|\dot{G}(f)|^2}{E_s} \, df \]  \hfill (9.23)

They are found to be:

\[ \Delta t^2 = \int_{-\infty}^{\infty} \frac{t |\dot{g}(t)|^2}{E_s} \, dt - \Delta t^2 \]  \hfill (9.24)

\[ \Delta f^2 = \frac{1}{4\pi^2 E_s} \int_{-\infty}^{\infty} \frac{d\phi(t)}{dt} w(t)^2 \, dt - \frac{j}{4\pi^2 E_s} \int_{-\infty}^{\infty} \frac{d\phi(t)}{dt} w(t) \, dt, \]  \hfill (9.25)

The latter result means that the bandwidth is completely defined by the amplitude variation \( \frac{d\phi(t)}{dt} \) as well as phase variations \( \frac{d\phi(t)}{dt} \). If both, the amplitude and the phase are constant, such as the complex sinusoidal signal \( \exp(j 2\pi f_0 t) \) then the frequency bandwidth reduces to zero.

It can be shown that:

\[ \Delta t \cdot \Delta f \geq \frac{1}{4\pi}. \]  \hfill (9.26)

This result is known as the uncertainty principle and comes to show that the time-bandwidth (TB) product of a signal has a lower limit.
Figure 9.3: Illustration of the pulse compression and the influence of the pulse duration and bandwidth on the auto correlation function. The top row shows 3 real signals - two frequency modulated pulses and a one-cycle sine pulse. The duration of the FM pulses and the sine pulse is 30 \( \mu s \) and 1 \( \mu s \) respectively. The mean frequency \( \bar{f} \) of the signals is 1 MHz. The middle row shows the frequency density spectra. The bottom row shows the magnitude (envelope) of their autocorrelation functions.

9.1.3 Pulse compression

As seen in Section 9.1.1 the signal that determines the imaging abilities of the system is \( r_m(t) \), which is the output of the matched filter, and at the same time is the autocorrelation function of the transmitted pulse \( g(t) \). Rihaczek [47] reported an approximate equation for the half-power width of the main lobe:

\[
\Delta \tau = \frac{1}{4\Delta f}
\]  

(9.27)

There are different definitions for the duration of the pulse and its bandwidth. Qian and Chen [116] use the same definition for \( \Delta \), but because they work with angular frequency \( \omega \), their product is \( \Delta \cdot \Delta \omega \geq \frac{1}{2} \). Rihaczek [47] on the other hand defines the RMS (root mean square) duration and bandwidth as \( \delta = 2\pi\Delta \) and \( \beta = 2\pi\Delta f \), respectively. The uncertainty principle is then \( \beta \cdot \delta \geq \pi \).
On the other hand, the peak value of $r_m(t)$ being the value of the autocorrelation function at lag 0, is equal to the energy contained in the signal. The gain in the signal-to-noise ratio is given by:

$$GSNR = \frac{SNR_{out}}{SNR_{in}} = TB,$$

where $TB$ is the time-bandwidth product.

From these two results it is obvious that the duration of the pulse must be chosen according to the desired gain in signal-to-noise ratio. The bandwidth of the signal can be increased (See Eq. (9.25)) by applying either amplitude or phase modulation. Changing the phase does not affect the carried energy and therefore is the preferred approach. From (9.21) one will learn that any linear phase change only will change the mean frequency $f_0$, and therefore a non-linear modulation function for $\varphi(t)$ must be used. One set of signals are the frequency modulated signals, an example of which is shown in Figure 9.3. The two long signals are linear frequency modulated signals (chirps), which are defined as:

$$\dot{g}(t) = w(t)e^{j2\pi(f_0t + \beta t^2)}$$

where $\beta$ is a constant showing how fast the instantaneous frequency changes and $w(t)$ is a windowing function. All of the signals have a mean frequency $\bar{f}$ of 1 MHz. The chirp in the
Take 2 window functions Split them Combine them

Figure 9.5: Illustration of the filter design for weighting of the chirps.

The first column has RMS bandwidth $2\Delta f$ of 807 kHz. The chirp in the middle column and the sine pulse have RMS bandwidths around 2 MHz. The figure clearly shows that the energy in the signals is proportional to their length, and that the width of main lobe of the autocorrelation function is inversely proportional to the bandwidth. Using either a chirp or a single frequency RF pulse with the same RMS bandwidth results in the same spatial/temporal resolution, but the energy contained in the chirp is bigger. From (9.28) the GSNR of the matched filter for the sine pulse and the middle chirp in Figure 9.3 are:

$$GSNR_{\text{chirp}} = T_{\text{chirp}} \times B_{\text{chirp}} \approx 30 \cdot 10^{-6} \text{s} \times 2 \cdot 10^6 \frac{1}{\text{s}} = 17.78 \text{ dB}$$

$$GSNR_{\text{sine}} = T_{\text{sine}} \times B_{\text{sine}} \approx 1 \cdot 10^{-6} \text{s} \times 2 \cdot 10^6 \frac{1}{\text{s}} = 3.01 \text{ dB}$$

This result shows that using the linear chirp given in Figure 9.3 instead of the sinusoidal pulse increases the signal-to-noise ratio with 14.7 dB.

Because of the nearly rectangular shape of the spectrum, the envelope of the autocorrelation function of a chirp signal has side lobes, which in the literature are known as range side lobes. These range side lobes reduce the contrast of the image and their level must be decreased. This can be done by applying a windowing function $w(t)$ different than a rectangle [117, 118] as shown in Figure 9.4. The top row and middle rows show three different signals in time and frequency domain, respectively. The bottom row shows a comparison between the output of their correlation functions. The narrowest main-lobe of the autocorrelation function is achieved by the non-tapered chirp, and the lowest side lobe level by the Hanning weighted chirp. A good compromise is the chirp apodized with a flat-top$^2$ filter. A substantial work in designing such flat-top filters was done by my colleague Thanassis Misaridis [119, 120] in his Ph.D. work and the signals used further on in the dissertation are based on his design.

The window used for the signal in the right-most column of Figure 9.4 was made by combining a rectangular and a Kaiser window functions as shown in Figure 9.5. The discrete Kaiser

$^2$The term flat-top here is used to denote that most of the filter coefficients are equal to 1. Normally this term is used to denote filters with flat frequency response, which is not the case here.
9.1. Temporal coding

Figure 9.6: The output of a filter when there is a mismatch between the center frequency of the signal and the center frequency of the impulse response of the filter.

The output signal is a replica of the transmitted pulse. The design of the matched filter should also take into consideration the influence of the transducer bandwidth. Also, if some other optimization criteria are chosen, then the filter used for compression might deviate from the matched filter. This problem goes, however, beyond the scope of this work and will not be treated further.

Another assumption tacitly made to this moment was, that the received waveform had the same frequency content as the emitted one. The ultrasound energy, however, while propagating in the human body attenuates. The attenuation is frequency dependent and is bigger for higher frequencies. The mean frequency and bandwidth are therefore decreased. Because of the difference in the frequency content between the transmitted and received pulses, the matched filter is not any more matched. Figure 9.6 shows the real part of the output of a filter matched to a Gaussian pulse with a mean frequency $\bar{f} = f_0 = 5$ MHz and bandwidth $2\Delta f = 3.13$ MHz.

The input signal was a Gaussian pulse, whose mean frequency was varied. The left plot shows the response of the filter, when all the responses are normalized with respect to the highest output, which occurs at $\Delta f = 0$ and $\tau = 0$. The right plot shows the outputs, each of which was normalized by the highest value contained in it. It can be seen that both the amplitude and the shape of the output changes. Such an output is described by the ambiguity function.
9.1.4 The ambiguity function

The ambiguity function is defined \([47, 116]\) as:

\[
\chi(\tau, f_d) = \int_{-\infty}^{\infty} \hat{g}(t)\hat{g}^*(t-\tau)\exp(j2\pi f_d t) \, dt,
\]

(9.33)

It can be seen that when there is no difference in the frequency, i.e. \(f_d = 0\), the ambiguity function is the matched filter response. The value of the ambiguity function away from the origin shows the response of a filter which is mismatched by a delay \(\tau\) and frequency shift \(f_d\) relative to the transmitted signal. Of interest is only the behavior of the function when there is a frequency difference.

Figure 9.7: The ambiguity functions of a long (left) and short (right) pulses. The short pulse is well localized in time, while the long pulse is well localized in frequency.

Figure 9.8: Comparison between the ambiguity function of an RF pulse (left) and a chirp (right) with the same duration.

\(^3\)In some sources this formula is known as the narrow-band ambiguity function.
Consider Figure 9.7 which shows the properties of the ambiguity function. A short pulse is well localized in time, but not in frequency domain. The output of the frequency mismatched filter will produce a sharp peak for the short pulse. The long pulse, however will be entirely filtered out.

Figure 9.8 shows the ambiguity functions of an RF pulse with a single frequency carrier with $f_0 = 5 \text{ MHz}$ and a linear frequency modulated RF pulse ranging from $f_1 = 2$ to $f_2 = 8 \text{ MHz}$. The duration of both pulses is $T = 5 \mu s$. The linear FM modulated pulse has a narrow peak of the autocorrelation function inversely proportional to its bandwidth. The single carrier RF pulse has a significantly larger peak of the autocorrelation function proportional to the length of the pulse. A frequency mismatch during the filtration of the single carrier RF pulse does not change the position of the maximum, but its amplitude gets smaller and after frequency mismatch $\Delta f = f_d = 1 \text{ MHz}$ the signal is completely filtered out. A frequency mismatch in the filtration of the linear FM pulse results in a time shift of the peak of the autocorrelation function. The shape of the autocorrelation function remains, however, unchanged making the use of linear FM chirps appealing in the presence of the Doppler shift as it will be discussed in Chapter 10.

### 9.1.5 Binary codes

The binary codes have been widely investigated for ultrasound imaging, and though not used in this work they are worth noting. Several types of codes such as Barker codes [121, 122], and m-sequences [123, 124] have been investigated. However, the most promising seem to be the codes based on Golay pairs which have been tried by a number of groups around the world. In 1979 Takeuchi proposed and described a system using Golay codes [125, 126]. Mayer and colleagues have successfully applied coded excitation based on Golay codes in synthetic aperture imaging [57]. More recently a group from General Electric has been investigating the use of such codes together with spatial Hadamard encoding [127, 128]. The idea behind the use of Golay codes is illustrated in Figure 9.9. Two binary codes are necessary: $A$ and $B$. Each
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Figure 9.10: Creating waveforms using complementary Golay codes. From top to bottom: the golay pair; base-band signal; RF signal.

of the binary codes has an autocorrelation function, whose side lobes are equal in magnitude and opposite in sign to the corresponding side lobes of the other code. Thus the sum of the autocorrelation functions results in cancellation of the side lobes as shown in the figure. A good description of what Golay sequences are, and how to construct them is given by Đoković [129].

The codes being binary are encoded as a waveform using phased modulation. Zero corresponds to an initial phase 0, and 1 to an initial phase $\pi$. The creation of the binary waveform for a 16 bit Golay pair is illustrated in Figure 9.10. The baseband signal contains rectangular pulses with amplitude $\pm 1$, depending on the phase. The RF signal is a 5 MHz phase modulated signal. For each of the bits one period at the carrier frequency is used.

For imaging purposes the use of Golay pairs means to transmit two times in the same direction—the first time using sequence $A$, the second time sequence $B$. The received signals are matched-filter with the respective matched filters for each of the transmitted pulses. The beamformed signals from the two emissions are summed together prior to envelope detection and display.

Figure 9.11 shows the performance of the Golay pairs illustrated in Figure 9.10. With black line is given the envelope of the sum of the autocorrelations functions of the RF signals assuming a transducer with impulse response $\delta(t)$. The blue line shows the envelope of the same transmitted pulse, but in the presence of a transducer with two-sided fractional bandwidth of 65% around 5 MHz. The main lobe gets wider in the presence of a transducer. The red line shows what would happen if only a binary waveform at baseband is transmitted and used for matched filter. The main lobe gets wider and the side-lobes get higher.
9.2 Spatial coding

9.2.1 Principle of spatial coding

The spatial encoding was first suggested by Silverstein in [130] for calibration of satellite antennas. Later on, the same year, Chiao, Thomas and Silverstein [3], suggested this method to be used for ultrasound imaging.

The idea behind it is very simple. In synthetic transmit aperture \( N_{xmt} \) emissions are necessary to create an image. Every emission is done by using a single virtual source element. The transmitted energy is proportional to the number of elements forming the virtual source, but there is a limit to how many these elements can be. One way to increase the transmitted power is instead of transmitting only by a single virtual source, to transmit with all of the virtual sources at the same time. The beamforming is based on the round trip from the virtual source to the receive elements. It is therefore necessary to distinguish which echo from the emission of which source is created. This can be done by a set of mutually orthogonal signals. Such codes haven’t been found yet. Another way is to send \( N_{xmt} \) times, every time with all of the \( N_{xmt} \) virtual sources with with some kind of encoding applied on the apodization coefficients.

Let the signal transmitted by a single source \( i \) be:

\[
g_i(t) = q_i \cdot g(t), \quad 1 \leq i \leq N_{xmt},
\]

(9.34)

where \( g(t) \) is a basic waveform and \( q_i \) is an encoding coefficient.

Assuming a linear propagation medium, the signal \( r_j(t) \) received by the \( j \)th element can be expressed as:

\[
r_j(t) = \sum_{i=1}^{N_{xmt}} q_i \cdot r_{ij}(t),
\]

(9.35)

where \( r_{ij}(t) \) would be the signal received by element \( j \), if the emission was done only by the \( i \)th source.
From Eq. (5.8) on page 58:

\[ L_i(t) = \sum_{j=1}^{N_{dc}} a_{ij}(t) r_{ij}(t - \tau_{ij}(t)) \]

it can be seen that the components \( r_{ij}(t) \) must be found in order to beamform the signal. The received signals can be expressed in a matrix form:

\[
\begin{bmatrix}
  r_j^{(1)}(t) \\
  r_j^{(2)}(t) \\
  \vdots \\
  r_j^{(N_{mt})}(t)
\end{bmatrix} =
\begin{bmatrix}
  q_1^{(1)} & q_2^{(1)} & \cdots & q_{N_{mt}}^{(1)} \\
  q_1^{(2)} & q_2^{(2)} & \cdots & q_{N_{mt}}^{(2)} \\
  \vdots & \vdots & \ddots & \vdots \\
  q_1^{(N_{mt})} & q_2^{(N_{mt})} & \cdots & q_{N_{mt}}^{(N_{mt})}
\end{bmatrix}
\begin{bmatrix}
  r_{1j}(t) \\
  r_{2j}(t) \\
  \vdots \\
  r_{N_{mt}j}(t)
\end{bmatrix}
\]

(9.36)

where the superscript \( (k) \), \( 1 \leq k \leq N_{mt} \) is the number of the emission, \( q_i^{(k)} \) is the encoding coefficient applied in transmit on the transmitting source \( i \), and \( r_j^{(k)}(t) \) is the signal received by the \( j \)th element. It was tacitly assumed that the tissue is stationary, and the building components of the signals \( r_j^{(k)} \) are the same for all the emissions:

\[
r_{ij}^{(1)}(t) = r_{ij}^{(2)} = \cdots = r_{ij}^{(k)}(t) = r_{ij}(t)
\]

(9.37)

More compactly, the equation can be written in a vector form as:

\[
\vec{r}_j(t) = Q\vec{r}_{ij}(t), \tag{9.38}
\]

where \( Q \) is the encoding matrix. The components \( r_{ij}(t) \) can be found by solving the above equation:

\[
\vec{r}_{ij}(t) = Q^{-1}\vec{r}_j(t) \tag{9.39}
\]

Every invertible matrix \( Q \) can be used for encoding and decoding. In Silverstein’s work [130] proves are given that the class of equal amplitude renormalized unitary matrices such as the 2-D DFT and Hadamard matrices are optimal for encoding of coherent signals. In the work of Chiao and colleagues [3], the choice was on using bipolar Hadamard matrices. Their advantage is the use of only two coefficients: +1, and -1. The Hadamard matrix is given by its order \( N \) (for more of the properties of the Hadamard matrices and the Hadamard transform, one can see [131], Section 3.5.2). The lowest order is \( N = 2 \), and the order can be only an even number. The Hadamard matrix of order 2 is defined as:

\[
H_2 = \begin{bmatrix}
  1 & 1 \\
  1 & -1
\end{bmatrix} \tag{9.40}
\]

The matrix of order \( 2N \) is recursively defined by the matrix of order \( N \) as:

\[
H_{2N} = H_2 \otimes H_N \tag{9.41}
\]

\[
= \begin{bmatrix}
  H_N & H_N \\
  H_N & -H_N
\end{bmatrix}, \tag{9.42}
\]

where \( \otimes \) is a Kronecker product. The inverse of the Hadamard matrix is the matrix itself:

\[
H_N^{-1} = \frac{1}{N}H_N, \tag{9.43}
\]

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9.2. Spatial coding

Figure 9.12: Spatially encoded transmits using 4 transmit elements.

which makes the decoding quite easy. The advantages of the Hadamard matrix include: (1) simple encoding mechanism - change in the polarity of the signals, (2) no multiplications are involved in the decoding stage, (3) there exist fast algorithms for the Hadamard transform significantly reducing the number of operations.

Figure 9.12 shows how the spatial encoding is applied for a Hadamard matrix of order 4. The elements are evenly distributed across the whole aperture. These can be either single elements, or a group of elements creating a virtual source, whose apodization coefficients are multiplied with the encoding values of the Hadamard matrix.

Let’s consider an example of coding and decoding using a 2 \( \times \) 2 Hadamard matrix. The transmitting elements have indices 1 and \( N_{xmt} \). The signal received by a single element \( j \) is:

\[
\begin{align*}
    r_j^{(1)}(t) &= r_{1j}(t) + r_{N_{xmt}j}(t) \\
    r_j^{(2)}(t) &= r_{1j}(t) - r_{N_{xmt}j}(t)
\end{align*}
\]  

(9.44)

Getting the sum and the difference of the signals \( r_j^{(1)}(t) \) and \( r_j^{(2)}(t) \) one gets:

\[
\begin{align*}
    r_j^{(1)}(t) + r_j^{(2)}(t) &= 2r_{1j}(t) \\
    r_j^{(1)}(t) - r_j^{(2)}(t) &= 2r_{N_{xmt}j}(t)
\end{align*}
\]  

(9.45)  

(9.46)

The resulting components can be directly plugged into the beamformation unit.

9.2.2 Gain in signal-to-noise ratio

The gain in signal to noise ratio in a synthetic transmit aperture system is proportional to the square root of the number of receive elements \( N_{rcv} \) and the number of virtual transmit sources

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Figure 9.13: The gain in pressure. The plot shows the ratio of the maximum of the amplitude of the generated pressure using a single element emission and a 4 elements Hadamard encoded emission. The plot is obtained by simulation in Field II. The left image shows the spatial distribution and the right shows the magnitude.

\[
G_{SNR_{STA}} \sim SNR_0 \sqrt{N_{xmt}N_{rcv}},
\]

where \(SNR_0\) is the signal-to-noise ratio of a single element. This situation is improved by the use of spatial Hadamard encoding. Consider again the case of only two emissions:

\[
\begin{align*}
  r_j^{(1)}(t) &= r_1j(t) + r_{N_{xmt}j}(t) + n^{(1)}(t) \\
  r_j^{(2)}(t) &= r_1j(t) - r_{N_{xmt}j}(t) + n^{(2)}(t),
\end{align*}
\]

where \(n^{(1)}(t)\) and \(n^{(2)}(t)\) represent independent white noise received at the two emissions. Summing the equations to get the signal out, one gets:

\[
\hat{r}_{1j}(t) = r_1j(t) + \frac{1}{\sqrt{2}}n(t),
\]

where \(n(t)\) is noise. The signal to noise ratio of the decoded data set is \(10\log_{10} 2\) decibels better than the dataset obtained from transmitting by the two transmitters separately. The gain in signal-to-noise ratio according to Chiao and colleagues. [3] is:

\[
G_{SNR_{Hadamard\ STA}} \sim SNR_0 N_{xmt} \sqrt{N_{rcv}}.
\]

Assuming that the signal-to-noise ratio is inversely proportional to the transmitted energy it is intuitive that the more elements emit, the higher is the transmitted energy, and the bigger the signal-to-noise ratio is. Figure 9.13 shows the gain in transmitted pressure in decibels. This plot
9.3 Experimental phantom and clinical trials

The measurements, that will be presented in this section are in no case a complete study of the signal-to-noise ratio and its improvement when coded excitations are used. They are a mere demonstration that: (1) using codes gives better images with higher penetration depth, and (2) synthetic aperture can be used in a clinical situation. The measurements were done on a tissue mimicking phantom with a frequency dependent attenuation of 0.5 (dB/(cm MHz)), and on a soon-to-graduate Ph.D. student.

The measurements were carried out using RASMUS (see: Appendix G for more details). The transducer is a linear array transducer type 8804 by B/K Medical A/S, Gentofte, Denmark. The relevant parameters of the transducer and the system are listed in Table 9.1. In transmit up to 128 elements are available. The amplitude of the voltage applied on the transducer elements was 50 V. The maximum gain in receive was 40 dB.

The experiments were intended to try the following combinations of encodings: (1) no encoding, (2) only temporal encoding, (3) only spatial encoding, and (4) temporal and spatial encoding. The number of active elements \( N_{\text{act}} \) used to create a single virtual element for the different experiments is given in Table 9.2.

In the table \( N_{\text{cont}} \) gives the number of emissions used to create a single frame. The conventional pulse, denoted with the abbreviation “Sin” is a 3 cycles Hanning weighted sinusoid at the center frequency \( f_0 = 5 \text{ MHz} \). The duration of the FM pulse, denoted with “chirp” is \( T = 20 \mu s \), and the fractional bandwidth is 80 %. The number of elements when spatial encoding is used, varies such that there is a gap of at least one non-active element between two neighboring virtual sources. The use of spatial encoding is symbolized by the suffix “_had”.

The results from the phantom measurements are shown in Figure 9.14. The images are logarithmically compressed and have a 50 dB dynamic range. The number of emissions used to create each of the images is given in the titles of the plots. Consider the first row \( (N_{\text{cont}} = 4) \).

---

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Center frequency</td>
<td>( f_0 )</td>
<td>7.5</td>
<td>MHz</td>
</tr>
<tr>
<td>Sampling frequency</td>
<td>( f_s )</td>
<td>40</td>
<td>MHz</td>
</tr>
<tr>
<td>Pitch</td>
<td>( d_x )</td>
<td>208</td>
<td>( \mu \text{m} )</td>
</tr>
<tr>
<td>Element width</td>
<td>( w )</td>
<td>173</td>
<td>( \mu \text{m} )</td>
</tr>
<tr>
<td>Element height</td>
<td>( h )</td>
<td>4.5</td>
<td>mm</td>
</tr>
<tr>
<td>Elevation focus</td>
<td>( F )</td>
<td>25</td>
<td>mm</td>
</tr>
<tr>
<td>Fractional bandwidth</td>
<td>( B )</td>
<td>50</td>
<td>%</td>
</tr>
<tr>
<td>Number of transducer elements</td>
<td>( N_{\text{edc}} )</td>
<td>192</td>
<td>-</td>
</tr>
<tr>
<td>Number of receive channels</td>
<td>( N_{\text{rcv}} )</td>
<td>64</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 9.1: Some parameters of the transducer and the system.
Figure 9.14: B-mode images of a tissue mimicking phantom for different combinations of temporal and spatial encoding schemes, and number of transmissions.
9.4 Conclusion

The number of active elements $N_{act}$ used for the different experiments as a function of the number of emissions and the encoding scheme. sin means “conventional pulse”, chirp means “FM pulse”, and had means “using Hadamard encoding”. $N_{xmt}$ is the number of emissions per frame.

Table 9.2: The number of active elements $N_{act}$ used for the different experiments as a function of the number of emissions and the encoding scheme. sin means “conventional pulse”, chirp means “FM pulse”, and had means “using Hadamard encoding”. $N_{xmt}$ is the number of emissions per frame.

<table>
<thead>
<tr>
<th></th>
<th>$N_{xmt} = 4$</th>
<th>$N_{xmt} = 8$</th>
<th>$N_{xmt} = 16$</th>
<th>$N_{xmt} = 32$</th>
<th>$N_{xmt} = 64$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>chirp</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>sin_had</td>
<td>13</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>chirp_had</td>
<td>13</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>-</td>
</tr>
</tbody>
</table>

At depth $z$ of 95 mm, from $x = -35$ to $x = 0$ mm in axial direction, there are 5 point scatterers. All of them can be seen only on the scans, in which FM pulses were used in transmit. Consider the vertical row of 3 point scatterers, lying at a lateral position $x \approx 35$ mm. It can not be seen in sin_4, and can be best observed in had_chirp_4. The visibility of the images in ascending order is: sin_4, had_sin_4, chirp_4, and had_chirp_4. An increased number of emissions $N_{xmt}$ increases the image quality for all encoding schemes.

Figure 9.15 shows the results from scanning the carotid artery. The position of the transducer is slightly different for the different scan situations and the results are not directly comparable. The general conclusion from these images is that the synthetic aperture can be used in-vivo.

There is a possibility to use a combination of two different excitations, whose cross-correlation is small, and combine them with spatial and temporal encoding using a Hadamard matrix [127]. The transmit sequence is:

\[
\begin{align*}
A_1 & \quad A_2 \quad B_3 \quad B_4 \\
A_1 & \quad -A_2 \quad B_3 \quad -B_4,
\end{align*}
\]

where $A$ and $B$ are two orthogonal excitations. First, Hadamard decoding is used to obtain the signals $A_1 + B_3$ and $A_2 + B_4$ emanating from the different virtual sources. Then correlation is applied on the result to separate $A_1$ from $B_3$, and $A_2$ from $B_4$. This procedure makes it possible to use 4 virtual sources, and only two emissions. This, however, increases even more the complexity of the system. The author has no doubt that in the future more and more sophisticated encoding schemes will be utilized in the scanners.

9.4 Conclusion

This chapter presented the basics of using coded excitations for ultrasound imaging combined with spatial encoding. In this work they are used only to increase the signal-to-noise ratio, rather than decrease the number of emissions. These encoding schemes have been implemented and used in the experimental scanner RASMUS developed at CFU. B-mode images of phantom and in-vivo experiments were given. The general conclusion is that at low number of emissions the image quality is increased by using FM pulses and spatial Hadamard encoding.

The second part of the dissertation deals with blood flow estimation. The estimates are mostly influenced by noise, and therefore the use of coded excitations is a must.
Figure 9.15: B-mode images of a carotid artery for different combinations of temporal and spatial encoding schemes, and number of transmissions.