Fresnel diffraction between confocal spherical surfaces

Figure A.1: Confocal spherical surfaces. On the left the 3-D case is shown. On the right a cut in the \((y - z)\) plane is shown.

The ultrasound imaging is performed in the near field using focused transducers. The focusing can be either mechanical or electronic. In the latter case the focus can be changed as a function of depth. This case corresponds to the case of diffraction between two confocal spherical surfaces such as the shown in Figure A.1. While in [23] Goodman describes the result of such a diffraction, here the full derivation is given.

The surface described by \((x_0, y_0, z_0)\) radiates a continuous wave, and the field is studied along the spherical surface \((x_1, y_1, z_1)\). The center of each of the spheres \((x_0, y_0, z_0)\) and \((x_1, y_1, z_1)\) lies on the surface of the other. In the depicted case the surface of the sphere \((x_0, y_0, z_0)\) is tangent to the \((x,y)\) plane at point \((0,0,0)\). The surface \((x_1, y_1, z_1)\) is tangent to a plane parallel to the \((x,y)\) plane lying at a distance \(z_f\). The distance \(z_f\) is the focal depth. For simplicity consider only the two dimensional case depicted on the right side of Figure A.1. The two surfaces can be expressed as:

\[
\begin{align*}
  z_0 &= z_f - \sqrt{z_f^2 - y_0^2} \\
  z_1 &= \sqrt{z_f^2 - y_1^2}
\end{align*}
\]  

\[(A.1)\]
The distance \( r_{01} \) between the two points \((y_0, z_0)\) and \((y_1, z_1)\) is

\[
r_{01} = \sqrt{(y_1 - y_0)^2 + (z_1 - z_0)^2}
\]

\[
r_{01} = \sqrt{(y_1 - y_0)^2 + \left(z_f - \sqrt{z_f^2 - y_0^2} - \sqrt{z_f^2 - y_1^2}\right)^2}
\]  
(A.2)

A simplification of the terms yields:

\[
r_{01} = \sqrt{3z_f^2 - 2y_0y_1 - 2z_f\sqrt{z_f^2 - y_0^2} + 2\sqrt{z_f^2 - y_0^2}\sqrt{z_f^2 - y_1^2}}
\]  
(A.3)

Assuming that \( y_0^2/z_f^2 \ll 1 \) and \( y_1^2/z_f^2 \ll 1 \) the radicals are expanded into Taylor series giving:

\[
r_{01} \approx \sqrt{3z_f^2 - 2y_0y_1 - 2z_f\left(1 - \frac{y_0^2}{2z_f^2}\right)} - 2z_f^2\left(1 - \frac{y_1^2}{2z_f^2}\right) + 2z_f^2\left(1 - \frac{y_0^2}{2z_f^2}\right)\left(1 - \frac{y_1^2}{2z_f^2}\right)
\]  
(A.4)

\[
r_{01} \approx \frac{y_0y_1}{z_f} + \frac{y_0^2y_1^2}{4z_f^3} \approx 0
\]  
(A.5)

\[
r_{01} \approx z_f - \frac{y_0y_1}{z_f}
\]  
(A.6)

In order for (A.7) to be fulfilled either the transducer must be weakly focused \( \left(\frac{\text{max}(y_0)}{z_f} \ll 1\right) \) or the observation interval must be small \( \left(\frac{\text{max}(y_1)}{z_f} \ll 1\right) \). Thus, for weakly focused transducers (A.7) is valid for a bigger off-axis distance, and for strongly focused transducers this approximation is valid over a narrow region.

Provided that these conditions are met, and going back to the 3-D case one gets:

\[
r_{01} \approx z_f - \frac{x_0y_1}{z_f} - \frac{y_0y_1}{z_f}
\]  
(A.8)

Substituting (A.8) in the equation of the Huygens-Fresnel principle:

\[
\Phi(x_1, y_1; z) = \frac{z}{j\lambda} \int \Phi(x_0, y_0; 0) \frac{\exp\left(j\frac{2\pi}{\lambda} r_{01}\right)}{r_{01}^2} \, dx_0 \, dy_0
\]

one gets the following Fresnel approximation of the field distribution for the focal zone:

\[
\Phi(x_1, y_1; z_f) = e^{jkz_f} \int \int \Phi(x_0, y_0; 0) e^{-j\frac{2\pi}{\lambda x_0^2 + y_0^2}} \, dx_0 \, dy_0
\]  
(A.9)
This equation aside from the multipliers and scale factors, expresses the field observed on the right-hand spherical cap as the Fourier transform of the field on the left-hand spherical cap.

Since the image coordinates \((x_1, y_1)\) lie on a spherical surface, it is often more convenient to express the result in spherical coordinates, and then the radiation pattern becomes a function of the elevation and azimuth angles.